Bloch-Fourier Approach to Classical Homogenization Problems
by **Carlos Conca (*)**

In this mini-course we will go into the so-called Bloch-Fourier approach to understand homogenization of periodic structures. The aims are to state the main results with motivations, highlight various phenomena in the Fourier space and indicate possible gains over other methods. Main ideas will be given trying to avoid technicalities.

It is fair to say that ever since the publication of the book [4], there is a renewed and vigorous activity in homogenization problems which form an important area of Applied Mathematics. In order to tackle the key questions in homogenization, several methods have been devised and there is an enormous literature on the subject. Our goal in these series of lectures is to address certain basic questions which we consider to be fundamental and to present a way to answer them which can be classified inside category (B) below.

Homogenization methods can be broadly categorized as follows: (A) Physical space methods, (B) Fourier space methods, and (C) Phase space methods. There are many methods falling in the class (A) and let us mention some successful ones : Multiscale expansion [4], Multiscale convergence [1], [21], Method of oscillating test functions [19], [24], Compensated Compactness [20], Gamma Convergence [11] and so on. Though they have varying domains of applicability, they have some common characteristics. An insight into the kind of oscillations produced by the medium is gained and this information is used in some way or the other (ansatz, test functions, ...).

In order to study partial differential equations with variable coefficients, techniques of pseudo-differential operators and Fourier integral operators have been developed [15]. These phase space theories provide tools suitable to study the behaviour of in-homogeneous media which are qualitatively similar to homogeneous ones and their small perturbations. To deal with heterogeneities possessing qualitatively different properties, we need more sophisticated tools such as H-measures [23], micro-local defect measures [13], Wigner measures [14], [16] and their generalizations and refinements (see [18]).

In the middle ground between the classes (A) and (C) lie the methods based purely on Fourier techniques and these mini-course address them. In problems involving oscillations, it is natural to seek methods based on Fourier space and this general outlook led us to look for tools suitable to describe oscillations produced by heterogeneities in the Fourier space. My intention is to present some of them and illustrate how they can be effectively used to study classical homogenization problems.

Though there are many aspects of homogenization, we will confine ourselves to some of them which I consider to be fundamental. Using the tools of Fourier space, one simple objective here is to shed new light and offer an alternate way to view classical results. Since these tools are sharp, we will also be able to bring out certain new features and results which are not easily obtainable by other means. This should attract the attention of potential young homogenizers. Multiscale structure of the solution, if any, will be a consequence and not an apriori assumption. Passage to the limit is more direct and there is no need for sophisticated test functions. The way the test functions arise in this approach will also be pointed out.

The following points will be discussed throughout:

(i) Existence of the homogenization limit.

(ii) When the limit exists, it will provide zeroth order approximation to the solution and to the non-homogeneous medium. What about higher order approximations called correctors? In this context, we will introduce a new object called the Bloch approximation. While the notion of correctors for the solution is well-known, the corresponding notion for the medium appears to be new. This possibility is due to the Fourier approach that we follow.

(iii) It is a classical fact in homogenization that first order derivatives of the solution are bounded. Under what condition, do we have uniform estimates on the second order derivatives? This aspect will be considered in these lectures and appears not discussed elsewhere.

The mini-course is essentially based on author's publications with their collaborators (see, e.g. [2], [3], [6] – [10]) and also on some current on-going unpublished works. Of course, the compiled bibliography is vast but obviously far from being comprehensive.

(*) Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile and Centro de Modelamiento Matemático, UMI 2807 CNRS-UChile, Casilla 170/3 - Correo 3, Santiago (Chile).

References

[1] Allaire G. (1992) Homogenization and two-scale convergence, Siam J. Math. Anal. 23, pp. 1482-1518.

[2] Allaire G. & Conca C. (1998) Bloch-wave homogenization and spectral asymptotic analysis, J. Math. Pures Appl. 77, pp. 153-208.

[3] Allaire G. & Conca C. (1996) Bloch wave homogenization for a spectral problem in fluid-solid structures, Arch. Rational Mech. Anal. 135(3), pp. 197-257.

[4] Bensoussan A., Lions J.L. & Papanicolaou G. (1978) Asymptotic Analysis in Periodic Structures, North-Holland, Amsterdam.

[5] Bloch F. (1928) Über die quantenmechanik der electronen in kristallgittern, Z. Phys. 52, pp. 555-600.

[6] Conca C., Orive R. & Vanninathan M. (2002) Bloch approximation in homogenization and applications, Siam J. Math. Anal. 33(5), pp. 1166-1198.

[7] Conca C., Orive R. & Vanninathan M. (2004) Bloch approximation in bounded domains, Asym. Anal. (in press)

[8] Conca C., Planchard J. & Vanninathan M. (1995) Fluids and Periodic Structures, Collection RAM 38, J. Wiley & Sons/Masson, New York/Paris.

[9] Conca C. & Vanninathan M. (1997) Homogenization of periodic structures via Bloch decomposition, Siam J. Appl. Math. 57(6), pp. 1639-1659.

[10] Conca C. & Vanninathan M. (2001) On uniform H2-estimates in periodic homogenization, Proc. Roy. Soc. Edinburgh 131A, pp. 499-517.

[11] Dal Maso G. (1993) An Introduction to Gamma-Convergence, Birkhäuser, Boston.

[12] Floquet G. (1883) Sur les équations difféerentielles linéaires à coefficients périodiques, Ann. Ecole Norm. Sér. 2 12, pp. 47–89.

[13] Gérard P. (1991) Microlocal defect measures, Commu. PDE 16 (11), pp. 1761-1794.

[14] Gérard P., Markowich P.A., Mauser N.J. & Poupaud F. (1997) Homogenization limits and Wigner transforms, Comm. Pure. Appl. Math. 50, pp. 321-377.

[15] Hörmander L. (1985) Analysis of Linear Partial Differential Operators III, Springer-Verlag, Berlin.

[16] Lions P.L. & Paul T. (1993) Sur les mesures de Wigner, Revista Math. Iberoamer. 9, pp. 553–618.
[17] Markowich P.A., Mauser N.J. & Poupaud F. (1994) A Wigner function approach to semiclassical limits: electrons in a periodic potential, J. Math. Phys. 35, pp. 1066–1094.
[18] Morgan R. & Babuˇska I. (1991) An approach for constructing families of homogenized equations for periodic media I and II, Siam J. Math. Anal. 2, pp. 1–15 & pp. 16–33.
[19] Murat F. (1977-78) H-Convergence, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, mimeographed notes. English translation: Murat F. & Tartar L. (1997) H-Convergence, in Topics in theMathematical Modelling of Composite Materials, A. Cherkaev & R. Kohn eds., Series Progress in Nonlinear Di_erential Equations and their Applications 31, Birkhäuser Verlag, Boston.
[20] Murat F. (1987) A survey on compensated compactness in Contributions to Modern Calculus of Variations, L. Cesari (Ed.), Pitman Research Notes in Mathematics, Series No. 148, pp. 145-183.
[21] Nguetseng G. (1989) A general convergence result for a functional related to the theory of homogenization, Siam J. Math. Anal. 20, pp. 608–623.
[22] Odeh F. & Keller J.B. (1964) Partial differential equations with periodic coefficients and Bloch waves in crystals, J. Math. Phys. 5, pp. 1499–1504.
[23] Tartar L. (1990) H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh 115 A, pp. 193–230.
[24] Tartar L. (1977) Problèmes d'homogénéisation dans les équations aux dérivées partielles, Cours Peccot au Collège de France, partially written in F. MURAT [19].
[25] Wilcox C. (1978) Theory of Bloch waves, J. Anal. Math. 33, pp. 146–167.